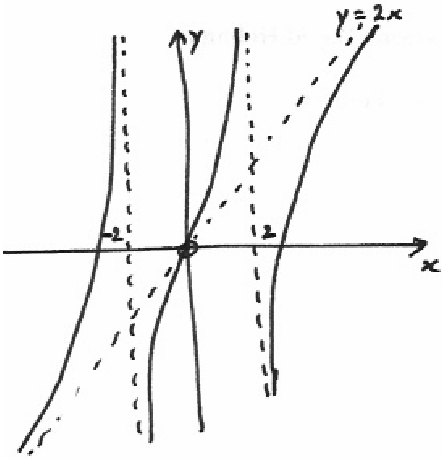
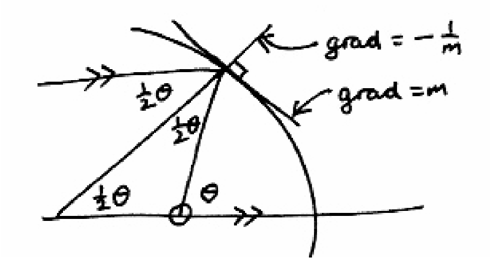
Step III, Hints and Answers
June 2006

1	$y=\frac{2x(x^2-5)}{x^2-4}$ $=2x-\frac{2x}{(x-2)(x+2)}$ Asymptotes are $y=2x$, $x=\pm 2$.  $\frac{dy}{dx}=2-\frac{2(x-2)(x+2)-4x^2}{(x-2)^2(x+2)^2}$ (or equivalent). Equation of the tangent at O is $y=\frac{5x}{2}$.
(i)	$3x(x^2-5)=(x^2-4)(x+3)$ $\Leftrightarrow \frac{2x(x^2-5)}{x^2-4}=\frac{2x}{3}+2 \quad (x\neq\pm 2)$ $y=\frac{2}{3}x+2$ cuts the sketched curve in three points, so three roots.
(ii)	$4x(x^2-5)=(x^2-4)(5x-2)$ $\Leftrightarrow \frac{2x(x^2-5)}{x^2-4}=\frac{5x}{2}-1 \quad (x\neq\pm 2)$. $y=\frac{5x}{2}-1$ passes through the intersection of $x=2$ and $y=2x$ and is parallel to $y=\frac{5x}{2}$ so just one root.
(iii)	$4x^2(x^2-5)^2=(x^2-4)^2(x^2+1)$ $\Leftrightarrow \frac{2x(x^2-5)}{x^2-4}=\pm\sqrt{(x^2+1)} \quad (x\neq\pm 2)$ $y=\pm\sqrt{(x^2+1)}$ has two branches with asymptotes $y=\pm x$, so there are six roots.
2 (i)	First "show" by change of variable $\theta=-\phi$ (say). Then

	$2I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{1+\sin\theta\sin 2\alpha} d\theta + \int_{-\pi/2}^{\pi/2} \frac{\cos^2\theta}{1-\sin\theta\sin 2\alpha} d\theta$ $= \int_{-\pi/2}^{\pi/2} \frac{2}{\sec^2\theta - \tan^2\theta\sin^2 2\alpha} d\theta$ and next "show" follows.
(ii)	$J = \sec 2\alpha \int_{-\pi/2}^{\pi/2} \frac{1}{1+(\cos 2\alpha\tan\theta)^2} \cos 2\alpha \sec^2\theta d\theta$ $= \sec 2\alpha \int_{-\pi/2}^{\pi/2} \frac{1}{1+u^2} du$ (since $\cos 2\alpha > 0$) $= \pi \sec 2\alpha$
(iii)	$I \sin^2 2\alpha + J\cos^2 2\alpha = \pi$. Result follows after use of (ii).
(iv)	In this case, $\cos 2\alpha < 0$, so $J = -\pi \sec 2\alpha$. Then $I = \frac{1}{2}\pi \operatorname{cosec}^2\alpha$
3 (i)	$\tan x$ is an odd function. Express both sides in terms of $\tan x$. From identity, substitute series and result follows by equating coefficients of powers of x.
(ii)	Show that $\cot x + \tan x = 2\operatorname{cosec} 2x$ and follow same method.
(iii)	Identity follows from $1+\cot^2 x = \operatorname{cosec}^2 x$. Equate coefficients to show that all coefficients for even n are zero, and $a_1 = 1, a_3 = \frac{1}{3}$.
4	Let $x = y$ and deduce first result. $2f(x) = f(2x)$ $\Rightarrow 2f'(x) = 2f'(2x)$ $\Rightarrow 2f''(x) = 4f''(2x)$ then put $x = 0$ to get $f(0) = 0, f''(0) = 0$. Similarly all higher order derivatives are zero, so by Maclaurin the most general function is cx, where c is a constant.
(i)	Use properties of logs to show that $G(x)+G(y) = G(x+y)$. Deduce that $g(x) = e^{cx}$.
(ii)	Show that $H(u)+H(v) = H(u+v)$ so $h(x) = c\ln x$.
(iii)	Let $T(x) = t(\tan x)$. Deduce that $t(x) = c\arctan x$.
5	There are essentially two different configurations, corresponding to clockwise and anticlockwise arrangements of α, β, γ taken in order. In what follows, $\omega = \frac{-1+\sqrt{3}}{2}$, the cube root of unity with modulus 1 and argument $\frac{2\pi}{3}$; $1+\omega+\omega^2 = 0$ (*) is assumed.

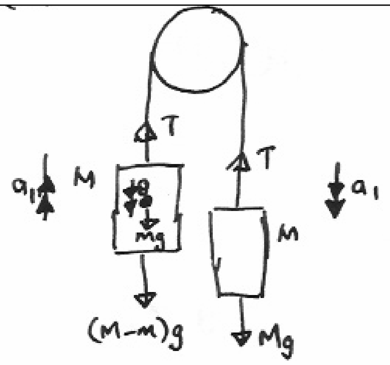
<table>
<tr><td></td><td>Then either $\beta-\gamma=\omega(\gamma-\alpha)$ and $\beta-\gamma=\omega^2(\gamma-\alpha)$ expresses equality of adjacent sides and the correct angle between them for each of the two cases; by SAS this establishes an equilateral triangle.
These two are equivalent to $[\beta-\gamma-\omega(\gamma-\alpha)][\beta-\gamma-\omega^2(\gamma-\alpha)]=0$.
The required form is an expanded version of this, using (*).
NB It is essential to be clear that the argument works both ways.
If α,β,γ are the roots of the equation given,
$-a=\alpha+\beta+\gamma, b=\alpha\beta+\beta\gamma+\gamma\alpha, c=-\alpha\beta\gamma$.
Then $a^2-3b=\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\gamma\alpha$
so $a^2-3b=0$ is equivalent to the expression in the first part.
Result follows.
$z\to pw$ is an enlargement combined with rotation, so object and image are similar. $pw\to pw+q$ is a translation so object and image are congruent.
Hence under the composition $z\to pw+q$ object and image are similar.
Result follows.
Aliter. Substitute $z=pw+q$ in the first equation, and simplify.
Compare coefficients to determine A and B in terms of a, b and c.
Then $a^2-3b=0\Rightarrow A^2-3B=0$, so result follows.</td></tr>
<tr><td>6</td><td>$x=r\cos\theta, y=r\sin\theta, r=r(\theta)$
$\Rightarrow \dfrac{dy}{dx}=\dfrac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}$
and result follows.


Gradient of the normal is $\tan\dfrac{\theta}{2}=t$, say. Then we have
$t=-\dfrac{\frac{dr}{d\theta}-r\tan\theta}{\frac{dr}{d\theta}\tan\theta+r}, \tan\theta=\dfrac{2t}{1-t^2}$
This reduces to</td></tr>
</table>

	$\dfrac{dr}{d\theta} = rt$ $\Rightarrow \ln r = \displaystyle\int \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} d\theta$ $\qquad = -2\ln\left[c\cos\dfrac{\theta}{2}\right]$ $\Rightarrow \dfrac{2}{c^2 r} = 1 + \quad \cos\theta$ (using $1+\cos\theta = 2\cos^2\dfrac{\theta}{2}$) This corresponds to the standard equation of a parabola in polars.
7 (i)	Express sinhx in terms of exponentials, factorise and solve to get $u = -e^x$ or $u = e^{-x}$ (or $-\cosh x \pm \sinh x$). Use both of these as equal to $\dfrac{dy}{dx}$ and integration to get alternative solutions $y = -e^{\pm x} + c.$ From the given conditions the particular integral is $y = 1 - e^{-x}$.
(ii)	Solve the quadratic as before to get either $u = \dfrac{-1 \pm \cosh y}{\sinh y}$ (or equivalent) $\Rightarrow \dfrac{dx}{dy} = \dfrac{\sinh y}{-1 \pm \cosh y}$ $\Rightarrow x = \ln(\cosh y - 1) + c_1$ or $x = -\ln(\cosh y + 1) + c_2$ Only the first can satisfy the conditions $x = 0, y = 0$; then we have $x = \ln\dfrac{2}{1+\cosh y}$ $\Rightarrow \cosh y = 2e^{-x} - 1$ This is undefined for $x > 0$. For $x \to -\infty \Rightarrow \cosh y \to \infty$, and there will be two branches, corresponding to $y \to \pm\infty$, as cosh is an even function. So $x \to -\infty \Rightarrow \cosh y \to \infty \Rightarrow y \to \infty \Rightarrow e^y \square 4e^{-x} \Rightarrow y = -x + \ln 4$ in one case, and similarly $y = x - \ln 4$ in the other.

8	Use (iv) with $f(x)\equiv 1, g(x)\equiv 1$ to show that $\Delta 1=0$. Use (iii) with $\lambda\equiv k, f(x)\equiv 1$ to show that $\Delta k=0$. By (iv), (i) $\Delta x^2=2x$; ditto $\Delta x^3=3x^2$. Now show $\Delta kx^n=knx^{n-1}$ by induction. Initial step is $\Delta k=0$; inductive hypothesis is that $\Delta kx^N=kNx^{N-1}$. Use (iii) and (iv) with hypothesis to show that $\Delta kx^{N+1}=k(N+1)x^N$. Now express any $P_k(x)$, a polynomial of degree k, as a sum of such powers, and so use (ii) to establish required result.
9	Take O as the zero level for potential energy. Then PE of bead at B is mgy; PE of particle at P is $mgr-mgl$. For perpetual equilibrium, the PE must have the same value in any position, in particular its value at H; result follows. Express equation shown in polar coordinates to get $r=\dfrac{2h}{1+\sin\theta}$ Differentiate and make $\dot{\theta}$ the subject so $\dot{\theta}=-\dfrac{\dot{r}(1+\sin\theta)^2}{2h\cos\theta}$. These two expressions give the desired result. By conservation of energy if PE is constant so is KE. Hence KE in a general position is equal to the initial value. That gives $V^2=\left(r\dot{\theta}\right)^2+2\dot{r}^2$ Speed of the particle at P is $\left\lvert\dot{r}\right\rvert$. Use the expressions for $V^2, \dot{\theta}$ to derive the required result.
10	Use conservation of angular momentum for the first result. Use conservation of energy to derive $v^2=\dfrac{k^2+a^2}{k^2}\Omega^2-(k^2+r^2)\omega^2$ and so by use of the first result and $v=-\dfrac{dr}{dt}$ second result follows. Now use $\omega=\dfrac{d\theta}{dt}$ and $\dfrac{dr}{d\theta}=\dfrac{dr}{dt}\Big/\dfrac{d\theta}{dt}$ and the two displayed result to derive the third. The suggested substitution transforms the third displayed equation to $\dfrac{du}{d\theta}=\sqrt{1+u^2}$. Invert and integrate to get the desired result. Hence $r=\dfrac{k}{\sinh(\theta+\alpha)}$. As $\theta\to\infty, r\to 0+$, but $r=0$ is impossible.

11



The equations of motion are

$$T-(M-m)g=(M-m)a_1$$

$$Mg-T=Ma_1$$

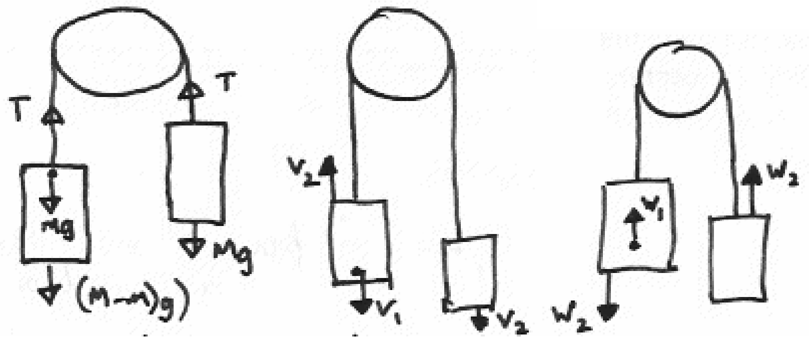
$$\Rightarrow a_1=\frac{mg}{2M-m}$$

Now consider relative motion of the tile with acceleration $(g+a_1)$.

If the time of the first stage is t_1, $s=ut+\frac{1}{2}at^2$ gives

$$t_1=\sqrt{\frac{(2M-m)h}{Mg}}$$

and then for the absolute motion of the tile $v=u+at$ gives the required final velocity.



The middle diagram shows the situation before the impact and the third after. The forces acting on the left-hand system (lift plus tile) are exactly the same as those on the right, so the changes in momentum must be equal in the first stage of the motion. Thus given that all is stationary initially

$$-(M-m)v_2+mv_1=Mv_2$$

$$\Rightarrow v_2=\frac{m}{2M-m}v_1=\alpha v_1 \text{ (*), say.}$$

In the collision, the equality of impulsive tensions given means that the change in momentum on one side equals change in momentum on the other. Hence we have

$$-Mw_2-Mv_2=-mw_1-mv_1+(M-m)(w_2+v_2)$$

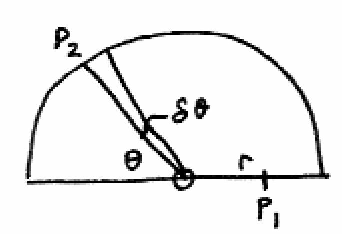
$$\Rightarrow w_2+v_2=\alpha(w_1+v_1)$$

Thus from the two last equations

$$w_2=\alpha w_1 \text{ (**)}.$$

Newton's experimental law and the two asterisked equations give

$$w_1=ev_1.$$

	Then the change of energy in the collision $\frac{1}{2}(2M-m)(v_2^2-w_2^2)+\frac{1}{2}m(v_1^2-w_1^2)$ simplifies to the required expression when the above relations are substituted. Loss of energy of a tile dropping to the floor of a fixed lift and bouncing would be just the same.
12	Model each tourist as trial with success probability ½. If X is the number of potential passengers $X \sim Bin(1024, \frac{1}{2})$, ie $N(512,16^2)$ approximately. Lost profit corresponds to $X > 480$. Hence if L is the loss, we have $E[L]=\sum_{k=1}^{32} kpr(X=480+k)+32pr(X>512)$ $=\sum_{k=1}^{32} k.\frac{1}{16}.\phi\left(-2+\frac{k}{16}\right)+16$ $\approx \int_0^{32} \frac{x}{16}\phi\left(-2+\frac{x}{16}\right)dx+16$ $=\int_0^{32} \frac{x}{16}\cdot\frac{1}{\sqrt{2\pi}}\cdot\exp\frac{(x-32)^2}{512}dx+16$ Now use substitution to show that this evaluates to $\frac{16}{\sqrt{2\pi}}\left(e^{-2}-1\right)+32\Phi(2)$. In the course of year the expectation is 50 times that figure, so that is the maximum tolerable licence fee.
13	 There are three cases to consider: (i) both on the circumference, (ii) P_1 on the diameter and P_2 on the circumference, and (iii) vice versa. For case (i), if P_1 lies in the arc $(\alpha,\alpha+\delta\alpha)$ P_2 lies in the arc $(\theta,\theta+\delta\theta)$, with probability $\frac{\delta\theta}{\pi+2}$, the area is $\frac{1}{2}\|r\|\sin\theta$. The expected area given P_1 lies in the arc $(\alpha,\alpha+\delta\alpha)$ is by integration $\frac{1}{\pi+2}$. For case (ii), if P_1 lies in $(r,r+\delta r)$ and P_2 lies in the arc $(\theta,\theta+\delta\theta)$, with probability $\frac{\delta\theta}{\pi+2}$, the area is $\frac{1}{2}\|r\|\sin\theta$. The expected area given P_1 lies in $(r,r+\delta r)$ from O is by integration $\frac{\|r\|}{\pi+2}$. Case (iii) is essentially the same as case (ii). Thus the expected area is $\int_0^{\pi}\frac{1}{\pi+2}.\frac{1}{\pi+2}d\alpha+2\int_{-1}^{1}\frac{\|r\|}{\pi+2}.\frac{1}{\pi+2}dr$ where the first integral corresponds to case (i) and the second to (ii) and (iii).

	This evaluates to the answer given.
14	$E[aX_1 + bX_2] = aE[X_1] + bE[X_2]$ $E[X_1X_2] = E[X_1]E[X_2]$ $E[P] = 2\mu_1 + 2\mu_2$ $E[P^2] = 4E[X_1^2] + 8E[X_1]E[X_2] + 4E[X_2^2]$ $E[X_1^2] = \mu_1^2 + \sigma_1^2$ $\text{var}[P] = 4(\sigma_1^2 + \sigma_2^2)$ The standard deviation is the square root of that expression. $E[A] = \mu_1\mu_2$ $E[A^2] = \mu_1^2\mu_2^2$ $\text{var}[A] = \sigma_1^2\mu_2^2 + \sigma_2^2\mu_1^2 + \sigma_1^2\sigma_2^2$ Again the standard deviation is the square root. Now find $\text{cov}[P, A] = 2\mu_2\sigma_1^2 + 2\mu_1\sigma_2^2$ This is not zero (as independence would imply) with given conditions. Similarly $\text{cov}[Z, A] = 2\sigma_1^2\mu_2 + 2\sigma_2^2\mu_1 - \alpha(\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2)$ That too is non-zero when α is not the excluded value. We consider the exceptional case with the given information. We have $\mu_1 = \mu_2 = 2, \sigma_1^2 = \sigma_2^2 = 1, \alpha = \frac{8}{9}$. Only three values of *A* are possible - 1, 3 and 9 - and they correspond to unique values of *Z*. Dependence can be shown by considering, for example, $pr\left(Z = \frac{28}{9}\right) = \frac{1}{4}, pr\left(Z = \frac{28}{9} \middle\| A = 3\right) = 0$.